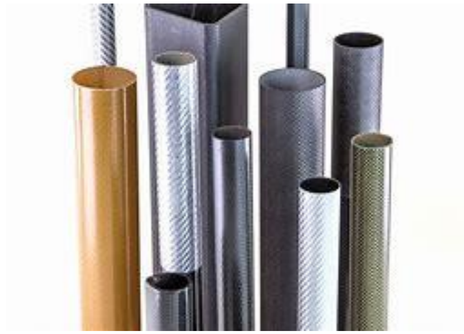
Composite Materials, PhD



Week 4

Macromechanical Analysis of a Lamina
Failure Theories

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Strength Failure Theories for an Angle Lamina

- The failure theories are generally based on the normal and shear strengths of a unidirectional lamina.

- In the case of a unidirectional lamina, the five strength parameters are:

  - ❑ Longitudinal tensile strength $(\sigma_1^T)_{ult}$
  - ❑ Longitudinal compressive strength $(\sigma_1^C)_{ult}$
  - ❑ Transverse tensile strength $(\sigma_2^T)_{ult}$
  - ❑ Transverse compressive strength $(\sigma_2^C)_{ult}$
  - ❑ In-plane shear strength $(\tau_{12})_{ult}$

Tsai-Hill Failure Theory

Based on the distortion energy theory, Tsai and Hill proposed that a lamina has failed if:

$$(G_2+G_3)\sigma_1^2+(G_1+G_3)\sigma_2^2+(G_1+G_2)\sigma_3^2-2G_3\sigma_1\sigma_2-2G_2\sigma_1\sigma_3$$
$$-2G_1\sigma_2\sigma_3+2G_4\tau_{23}^2+2G_5\tau_{13}^2+2G_6\tau_{12}^2<1$$

- This theory is based on the interaction failure theory.
- The components G_1 thru G_6 of the strength criteria depend on the strengths of a unidirectional lamina.

Components of Tsai-Hill Failure Theory

$$(G_2+G_3)\sigma_1^2+(G_1+G_3)\sigma_2^2+(G_1+G_2)\sigma_3^2-2G_3\sigma_1\sigma_2-2G_2\sigma_1\sigma_3$$
$$-2G_1\sigma_2\sigma_3+2G_4\tau_{23}^2+2G_5\tau_{13}^2+2G_6\tau_{12}^2<1$$

Apply $\sigma_1=(\sigma_1^T)_{ult}$, to a unidirectional lamina, then the lamina will fail. Hence, Equation reduces to:

$$(G_2+G_3)(\sigma_1^T)_{ult}^2=1$$

Components of Tsai-Hill Failure Theory

$$(G_2+G_3)\sigma_1^2+(G_1+G_3)\sigma_2^2+(G_1+G_2)\sigma_3^2-2G_3\sigma_1\sigma_2-2G_2\sigma_1\sigma_3$$
$$-2G_1\sigma_2\sigma_3+2G_4\tau_{23}^2+2G_5\tau_{13}^2+2G_6\tau_{12}^2<1$$

Apply $\sigma_2=(\sigma_2^T)_{ult}$, to a unidirectional lamina, then the lamina will fail.
Hence, Equation reduces to:

$$(G_1+G_3)(\sigma_2^T)_{ult}^2=1$$

Components of Tsai-Hill Failure Theory

$$(G_2+G_3)\sigma_1^2+(G_1+G_3)\sigma_2^2+(G_1+G_2)\sigma_3^2-2G_3\sigma_1\sigma_2-2G_2\sigma_1\sigma_3$$
$$-2G_1\sigma_2\sigma_3+2G_4\tau_{23}^2+2G_5\tau_{13}^2+2G_6\tau_{12}^2<1$$

Apply $\sigma_3=(\sigma_2^T)_{ult}$, to a unidirectional lamina, and assuming that the normal tensile failure strength is the same in direction (2) and (3), then the lamina will fail. Hence, Equation reduces to:

$$(G_1+G_2)(\sigma_2^T)_{ult}^2=1$$

Components of Tsai-Hill Failure Theory

$$(G_2+G_3)\sigma_1^2+(G_1+G_3)\sigma_2^2+(G_1+G_2)\sigma_3^2-2G_3\sigma_1\sigma_2-2G_2\sigma_1\sigma_3$$
$$-2G_1\sigma_2\sigma_3+2G_4\tau_{23}^2+2G_5\tau_{13}^2+2G_6\tau_{12}^2<1$$

Apply $\tau_{12}=(\tau_{12})_{ult}$ to a unidirectional lamina, then the lamina will fail. Hence, Equation reduces to

$$2G_6(\tau_{12})_{ult}^2=1$$

Components of Tsai-Hill Failure Theory

$$\left(G_2+G_3\right)\left(\sigma_1^T\right)_{ult}^2=1$$

$$\left(G_1+G_3\right)\left(\sigma_2^T\right)_{ult}^2=1$$

$$\left(G_1+G_2\right)\left(\sigma_2^T\right)_{ult}^2=1$$

$$2G_6\left(\tau_{12}\right)_{ult}^2=1$$

$$G_1=\frac{1}{2}\left(\frac{2}{\left[\left(\sigma_2^T\right)_{ult}\right]^2}-\frac{1}{\left[\left(\sigma_1^T\right)_{ult}\right]^2}\right)$$

$$G_2=\frac{1}{2}\left(\frac{1}{\left[\left(\sigma_1^T\right)_{ult}\right]^2}\right)$$

$$G_3=\frac{1}{2}\left(\frac{1}{\left[\left(\sigma_1^T\right)_{ult}\right]^2}\right)$$

$$G_6=\frac{1}{2}\left(\frac{1}{\left[\left(\tau_{12}\right)_{ult}\right]^2}\right)$$

Tsai-Hill Failure Theory – Plane Stress

Because the unidirectional lamina is assumed to be under plane stress - that is, $\sigma_3 = \tau_{31} = \tau_{23} = 0$,

$$(G_2+G_3)\sigma_1^2+(G_1+G_3)\sigma_2^2+(G_1+G_2)\sigma_3^2-2G_3\sigma_1\sigma_2-2G_2\sigma_1\sigma_3$$
$$-2G_1\sigma_2\sigma_3+2G_4\tau_{23}^2+2G_5\tau_{13}^2+2G_6\tau_{12}^2<1$$

$$\left[\frac{\sigma_1}{(\sigma_1^T)_{ult}}\right]^2-\left[\frac{\sigma_1\sigma_2}{(\sigma_1^T)_{ult}^2}\right]+\left[\frac{\sigma_2}{(\sigma_2^T)_{ult}}\right]^2+\left[\frac{\tau_{12}}{(\tau_{12})_{ult}}\right]^2<1$$

Tsai-Hill Failure Theory

- Unlike the Maximum Strain and Maximum Stress Failure Theories, the Tsai-Hill failure theory considers the interaction among the three unidirectional lamina strength parameter.

- The Tsai-Hill Failure Theory does not distinguish between the compressive and tensile strengths in its equation. This can result in underestimation of the maximum loads that can be applied when compared to other failure theories.

- Tsai-Hill Failure Theory underestimates the failure stress because the transverse strength of a unidirectional lamina is generally much less than its transverse compressive strength.

Example 4.1

Find the maximum value of S>0 if a stress of $\sigma_x = 2S, \sigma_y = -3S, \text{and } \tau_{xy} = 4S$ is applied to a 60° lamina of Graphite/Epoxy. Use Tsai-Hill Failure Theory. Use properties of a unidirectional Graphite/Epoxy lamina given in Table 2.1 of the textbook Mechanics of Composite Materials by Autar Kaw.

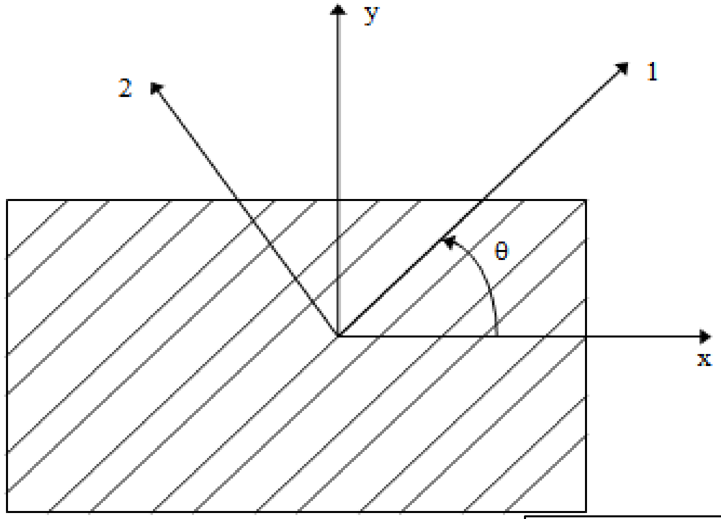

FIGURE 4.1
Off-axis loading in the x-direction

Solution

The stresses in the local axes are:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} 0.2500 & 0.7500 & 0.8660 \\ 0.7500 & 0.2500 & -0.8660 \\ -0.4330 & 0.4330 & -0.5000 \end{bmatrix} \begin{bmatrix} 2S \\ -3S \\ 4S \end{bmatrix}$$

$$= \begin{bmatrix} 0.1714 \times 10^{1} \\ -0.2714 \times 10^{1} \\ -0.4165 \times 10^{1} \end{bmatrix} S.$$

Example 4.2

$$\left[\frac{\sigma_1}{(\sigma_1^T)_{ult}}\right]^2 - \left[\frac{\sigma_1 \sigma_2}{(\sigma_1^T)_{ult}^2}\right] + \left[\frac{\sigma_2}{(\sigma_2^T)_{ult}}\right]^2 + \left[\frac{\tau_{12}}{(\tau_{12})_{ult}}\right]^2 < 1$$

$$\left(\frac{1.714S}{1500\times 10^6}\right)^2 - \left(\frac{1.714S}{1500\times 10^6}\right)\left(\frac{-2.714S}{1500\times 10^6}\right) + \left(\frac{-2.714S}{40\times 10^6}\right)^2 + \left(\frac{-4.165S}{68\times 10^6}\right)^2 < 1$$

$$S < 10.94\ MPa$$

Modified Tsai-Hill Failure Theory

$SR = 10.94$ (Tsai–Hill failure theory)

$SR = 16.33$ (maximum stress failure theory)

$SR = 16.33$ (maximum strain failure theory)

$$\left[\frac{\sigma_1}{X_1}\right]^2 - \left[\left(\frac{\sigma_1}{X_2}\right)\left(\frac{\sigma_2}{X_2}\right)\right] + \left[\frac{\sigma_2}{Y}\right]^2 + \left[\frac{\tau_{12}}{S}\right]^2 < 1$$

$$X_1 = \left(\sigma_1^T\right)_{ult}, if\ \sigma_1 > 0$$
$$= \left(\sigma_1^C\right)_{ult}, if\ \sigma_1 < 0$$

$$X_2 = \left(\sigma_1^T\right)_{ult}, if\ \sigma_2 > 0$$
$$= \left(\sigma_1^C\right)_{ult}, if\ \sigma_2 < 0$$

$$Y = \left(\sigma_2^T\right)_{ult}, if\ \sigma_2 > 0$$
$$= \left(\sigma_2^C\right)_{ult}, if\ \sigma_2 < 0$$

$$S = \left(\tau_{12}\right)_{ult}$$

Tsai-Wu Failure Theory

- Tsai-Wu applied the failure theory to a lamina in plane stress. A lamina is considered to be failed if:

$$H_1\sigma_1 + H_2\sigma_2 + H_6\tau_{12} + H_{11}\sigma_1^2 + H_{22}\sigma_2^2 + H_{66}\tau_{12}^2 + 2H_{12}\sigma_1\sigma_2 < 1$$

is violated. This failure theory is more general than the Tsai-Hill failure theory because it distinguishes between the compressive and tensile strengths of a lamina.

- The components $H_1 - H_{66}$ of the failure theory are found using the five strength parameters of a unidirectional lamina.

Components of Tsai-Wu Fail

a) Apply $\sigma_1 = \left(\sigma_1^T\right)_{ult}, \sigma_2 = 0, \tau_{12} = 0$ to a unidirectional lamina, the lamina will fail. Equation (2.152) reduces to:

$$H_1\left(\sigma_1^T\right)_{ult} + H_{11}\left(\sigma_1^T\right)_{ult}^2 = 1.$$

b) Apply $\sigma_1 = -\left(\sigma_1^C\right)_{ult}, \sigma_2 = 0, \tau_{12} = 0$ to a unidirectional lamina, the lamina will fail. Equation (2.152) reduces to:

$$-H_1\left(\sigma_1^C\right)_{ult} + H_{11}\left(\sigma_1^C\right)_{ult}^2 = 1.$$

From Equations (2.153) and (2.154),

$$H_1 = \frac{1}{\left(\sigma_1^T\right)_{ult}} - \frac{1}{\left(\sigma_1^C\right)_{ult}},$$

$$H_{11} = \frac{1}{\left(\sigma_1^T\right)_{ult}\left(\sigma_1^C\right)_{ult}},$$

Components of Tsai-Wu Fail

c) Apply $\sigma_1 = 0, \sigma_2 = \left(\sigma_2^T\right)_{ult}, \tau_{12} = 0$ to a unidirectional lamina, the lamina will fail. Equation (2.152) reduces to

$$H_2\left(\sigma_2^T\right)_{ult} + H_{22}\left(\sigma_2^T\right)_{ult}^2 = 1.$$

d) Apply $\sigma_1 = 0, \sigma_2 = -\left(\sigma_2^C\right)_{ult}, \tau_{12} = 0$ to a unidirectional lamina, the lamina will fail. Equation (2.152) reduces to:

$$-H_2\left(\sigma_2^C\right)_{ult} + H_{22}\left(\sigma_2^C\right)_{ult}^2 = 1.$$

From Equations (2.157) and (2.158):

$$H_2 = \frac{1}{\left(\sigma_2^T\right)_{ult}} - \frac{1}{\left(\sigma_2^C\right)_{ult}},$$

$$H_{22} = \frac{1}{\left(\sigma_2^T\right)_{ult}\left(\sigma_2^C\right)_{ult}}.$$

Components of Tsai-Wu Fail

e) Apply $\sigma_1 = 0, \sigma_2 = 0, \tau_{12} = (\tau_{12})_{ult}$ to a unidirectional lamina, the lamina will fail. Equation (2.152) reduces to:

$$H_6(\tau_{12})_{ult} + H_{66}(\tau_{12})_{ult}^2 = 1.$$

f) Apply $\sigma_1 = 0, \sigma_2 = 0, \tau_{12} = -(\tau_{12})_{ult}$ to a unidirectional lamina, the lamina will fail. Equation (2.152) reduces to:

$$-H_6(\tau_{12})_{ult} + H_{66}(\tau_{12})_{ult}^2 = 1.$$

From Equations (2.157) and (2.158),

$$H_6 = 0,$$

$$H_{66} = \frac{1}{(\tau_{12})_{ult}^2}.$$

Determination of $\mathbf{H_{12}}$

Apply equal tensile loads along the two material axes in a unidirectional composite. If

$\sigma_x = \sigma_y = \sigma$ $\tau_{xy} = 0$, is the load at which the lamina fails, then:

$$(H_1 + H_2)\sigma + (H_{11} + H_{22} + 2H_{12})\sigma^2 = 1.$$

The solution of the Equation (2.165) gives:

$$H_{12} = \frac{1}{2\sigma^2}\left[1 - (H_1 + H_2)\sigma - (H_{11} + H_{22})\sigma^2\right].$$

$$H_1\sigma_1 + H_2\sigma_2 + H_6\tau_{12} + H_{11}\sigma_1^2 + H_{22}\sigma_2^2 + H_{66}\tau_{12}^2 + 2H_{12}\sigma_1\sigma_2 < 1$$

Determination of $\mathbf{H_{12}}$

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Take a 45^0 lamina under uniaxial tension σ_x . The stress σ_x at failure is noted.

If this stress is $\sigma_x = 0$ then using Equation (2.94), the local stresses at failure are:

$$\sigma_1 = \frac{\sigma}{2},$$

$$\sigma_2 = \frac{\sigma}{2},$$

$$\tau_{12} = -\frac{\sigma}{2}.$$

Substituting the above local stresses in Equation (2.152):

$$(H_1 + H_2)\frac{\sigma}{2} + \frac{\sigma^2}{4}(H_{11} + H_{22} + H_{66} + 2H_{12}) = 1,$$

$$H_{12} = \frac{2}{\sigma^2} - \frac{(H_1 + H_2)}{\sigma} - \frac{1}{2}(H_{11} + H_{22} + H_{66}).$$

$$H_1\sigma_1 + H_2\sigma_2 + H_6\tau_{12} + H_{11}\sigma_1^2 + H_{22}\sigma_2^2 + H_{66}\tau_{12}^2 + 2H_{12}\sigma_1\sigma_2 < 1$$

Empirical Models of $\mathbf{H_{12}}$

$$H_{12} = -\frac{1}{2(\sigma_1^T)_{ult}^2}$$

as per Tsai-Hill failure theory[8]

$$H_{12} = -\frac{1}{2(\sigma_1^T)_{ult}(\sigma_1^C)_{ult}}$$

as per Hoffman criterion[10]

$$H_{12} = -\frac{1}{2}\sqrt{\frac{1}{(\sigma_1^T)_{ult}\,(\sigma_1^C)_{ult}\,(\sigma_2^T)_{ult}\,(\sigma_2^C)_{ult}}}$$

as per Mises-Hencky criterion[11]

Example 4.3

Find the maximum value of $S > 0$ if a stress $\sigma_x = 2S, \sigma_y = -3S$ and $\tau_{xy} = 4S$ are applied to a 60^0 lamina of Graphite/Epoxy. Use Tsai-Wu failure theory. Use the properties of a unidirectional Graphite/Epoxy lamina from Table 2.1.

Example 4.3

- Using Equation (2.94), the stresses in the local axes are:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} 0.2500 & 0.7500 & 0.8660 \\ 0.7500 & 0.2500 & -0.8660 \\ -0.4330 & 0.4330 & -0.5000 \end{bmatrix} \begin{bmatrix} 2S \\ -3S \\ 4S \end{bmatrix}$$

$$= \begin{bmatrix} 0.1714 \times 10^{1} \\ -0.2714 \times 10^{1} \\ -0.4165 \times 10^{1} \end{bmatrix} S.$$

H_{12}=

Example 4.3

$$H_1 = \frac{1}{1500\times10^6} - \frac{1}{1500\times10^6} = 0 \ \ Pa^{-1},$$

$$H_2 = \frac{1}{40\times10^6} - \frac{1}{246\times10^6} = 2.093\times10^{-8} \ \ Pa^{-1},$$

$$H_6 = 0 \ \ Pa^{-1},$$

$$H_{11} = \frac{1}{(1500\times 10^6)(1500\times10^6)} = 4.4444\times10^{-19}\, Pa^{-2},$$

$$H_{22} = \frac{1}{(40\times 10^6)(246\times10^6)} = 1.0162\times10^{-16}\, Pa^{-2},$$

$$H_{66} = \frac{1}{(68\times10^6)^2} = 2.1626\times10^{-16}\, Pa^{-2},$$

$$H_{12} = -0.5\left[\left(4.4444\times10^{-19}\right)\left(1.0162\times10^{-16}\right)\right]^{\frac{1}{2}} = -3.360\times10^{-18}\, Pa^{-2}.$$

Example 4.3

Substituting these values in Equation (2.152), we obtain:

$$(0)(1.714S)+(2.093\times 10^{-8})(-2.714S)$$
$$+(0)(-4.165S)+(4.4444\times 10^{-19})(1.714S)^2$$
$$+(1.0162\times 10^{-16})(-2.714S)^2+(2.1626\times 10^{-16})(4.165S)^2$$
$$+2(-3.360\times 10^{-18})(1.714S)(-2.714S)<1,$$

or

$$S<22.39MPa$$

Example 4.3

If one uses the other two empirical criteria for H_{12} as per Equation (2.171), one obtains:

$$S < 22.49 MPa \;\; for \; H_{12} = -\frac{1}{2(\sigma_1^T)_{ult}^2},$$

$$S < 22.49 MPa \; for \; H_{12} = -\frac{1}{2}\frac{1}{(\sigma_1^T)_{ult}(\sigma_1^C)_{ult}}.$$

Summarizing the four failure theories for the same stress-state, the value of S obtained is:

S = 16.33 (Maximum Stress failure theory),
= 16.33 (Maximum Strain failure theory),
= 10.94 (Tsai-Hill failure theory),
= 16.06 (Modified Tsai-Hill failure theory),
= 22.39 (Tsai-Wu failure theory).

Strength Failure Theories of an Angle Lamina

- The failure theories are generally based on the normal and shear strengths of a unidirectional lamina.
- An isotropic material generally has two strength parameters:

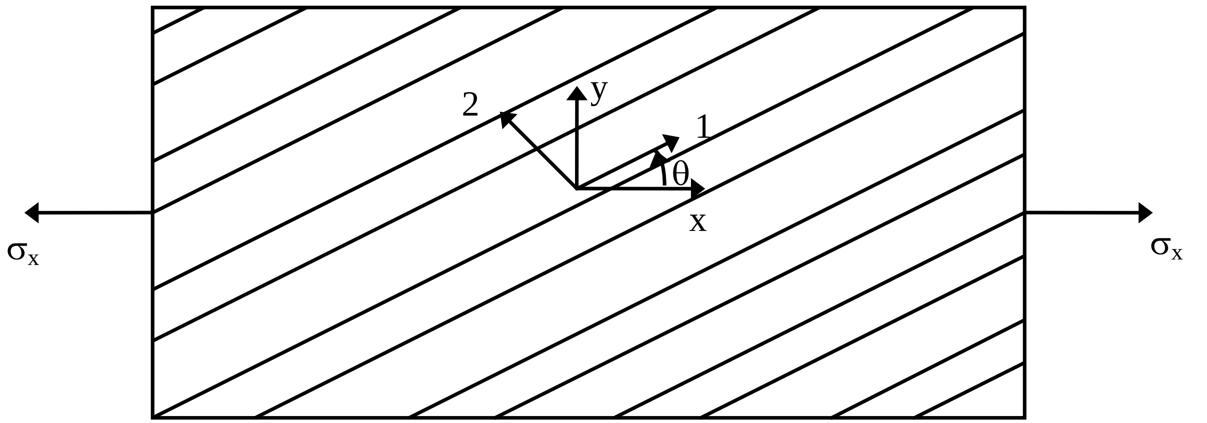
  normal strength and shear strength.

- In the case of a unidirectional lamina, the five strength parameters are:

❑ Longitudinal tensile strength $(\sigma_1^T)_{ult}$
❑ Longitudinal compressive strength $(\sigma_1^C)_{ult}$
❑ Transverse tensile strength $(\sigma_2^T)_{ult}$
❑ Transverse compressive strength $(\sigma_2^C)_{ult}$
❑ In-plane shear strength $(\tau_{12})_{ult}$

Experimental Results and Failure Theories

- Tsai and Wu compared the results from various failure theories to some experimental results. He considered an angle lamina subjected to a uniaxial load in the x-direction.

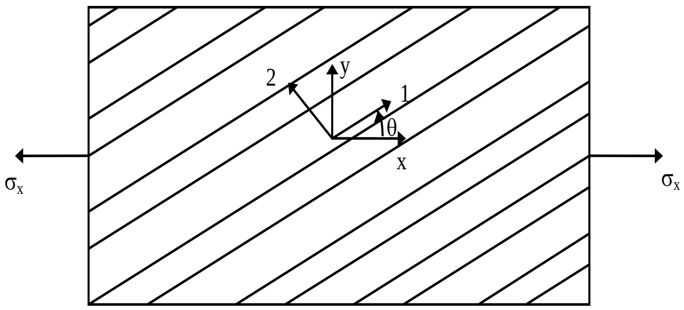


Experimental Results and Maximum Stress Failure Theory

$$-\left(\sigma_1^C\right)_{ult} \le \sigma_1 \le \left(\sigma_1^T\right)_{ult}$$

$$-\left(\sigma_2^C\right)_{ult} \le \sigma_2 \le \left(\sigma_2^T\right)_{ult}$$

$$-\left(\tau_{12}\right)_{ult} \le \tau_{12} \le \left(\tau_{12}\right)_{ult}$$



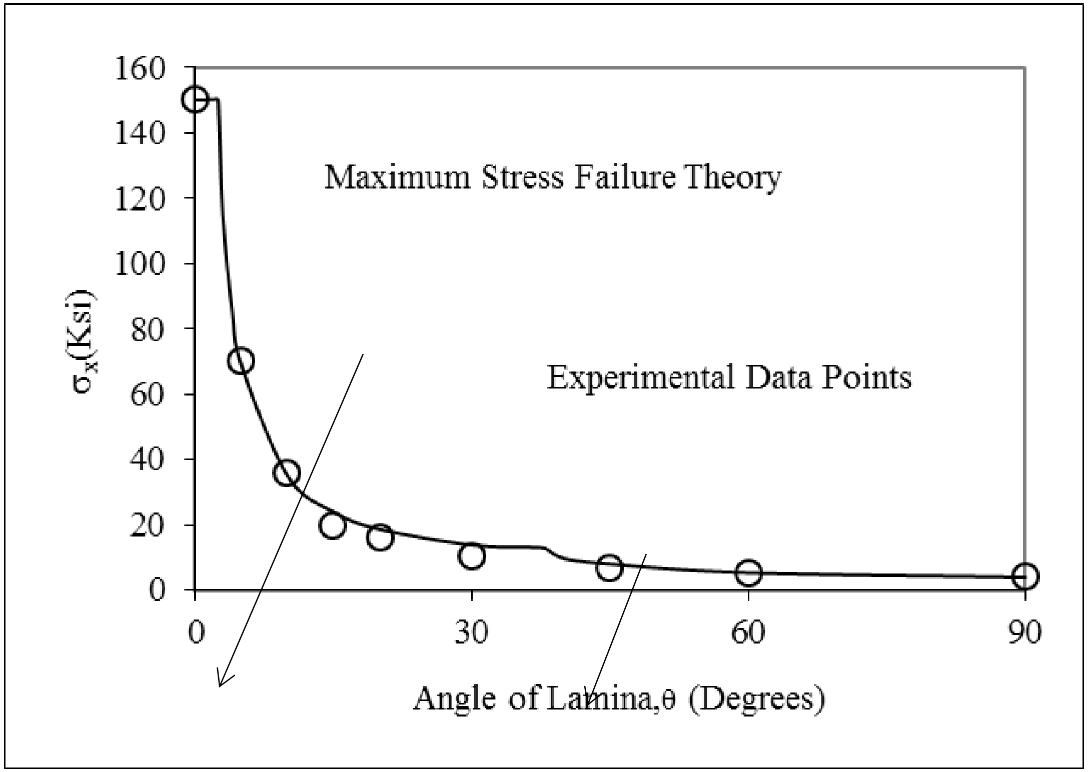


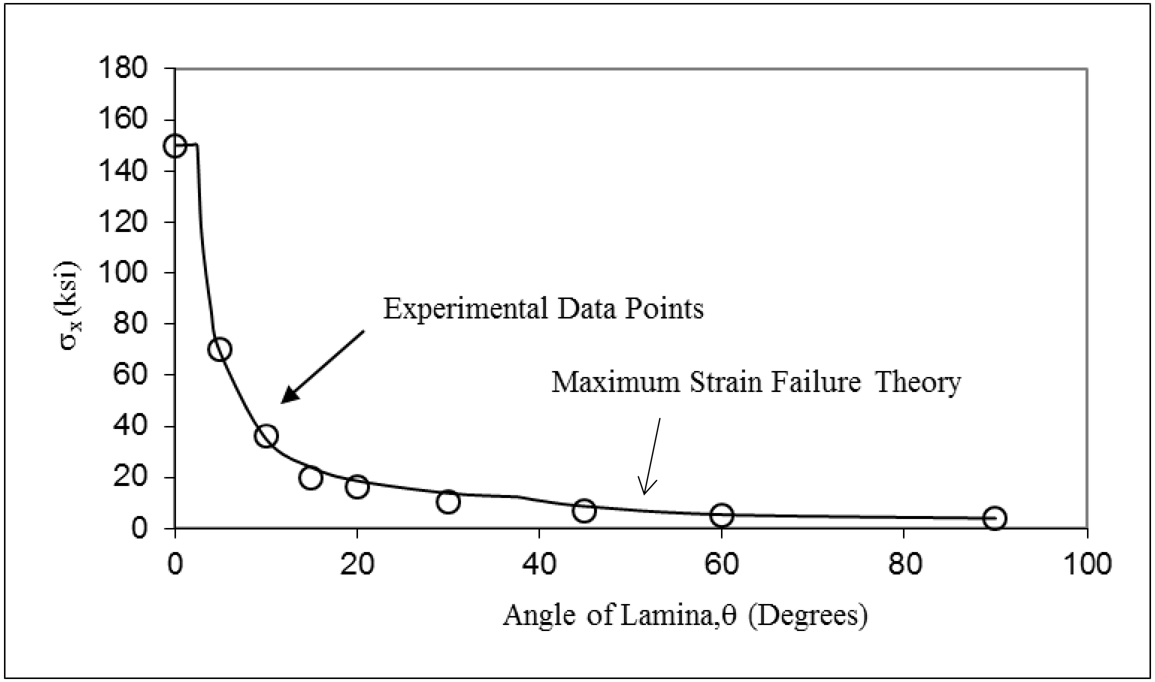
FIGURE 4.2
Maximum normal tensile stress in x-direction as function of angle of lamina using maximum stress failure theory

Experimental Results and Maximum Strain Failure Theory

$$-\left(\varepsilon_1^C\right)_{ult} < \varepsilon_1 < \left(\varepsilon_1^T\right)_{ult}$$

$$-\left(\varepsilon_2^C\right)_{ult} < \varepsilon_2 < \left(\varepsilon_2^T\right)_{ult}$$

$$-\left(\gamma_{12}\right)_{ult} < \gamma_{12} < \left(\gamma_{12}\right)_{ult}$$



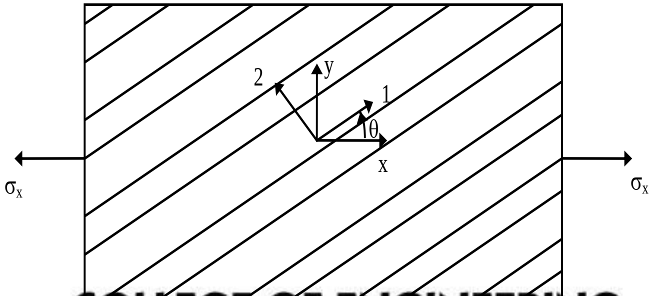
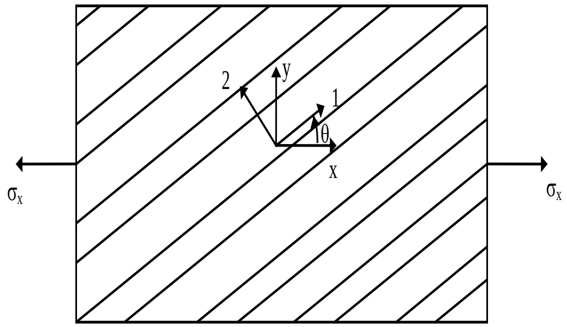


FIGURE 4.3
Maximum normal tensile stress in x-direction as function of angle of lamina using maximum Strain failure theory

Experimental Results and Tsai-Hill Failure Theory

$$\left[\frac{\sigma_1}{(\sigma_1^T)_{ult}}\right]^2 - \left[\frac{\sigma_1\sigma_2}{(\sigma_1^T)_{ult}^2}\right] + \left[\frac{\sigma_2}{(\sigma_2^T)_{ult}}\right]^2 + \left[\frac{\tau_{12}}{(\tau_{12})_{ult}}\right]^2 < 1$$



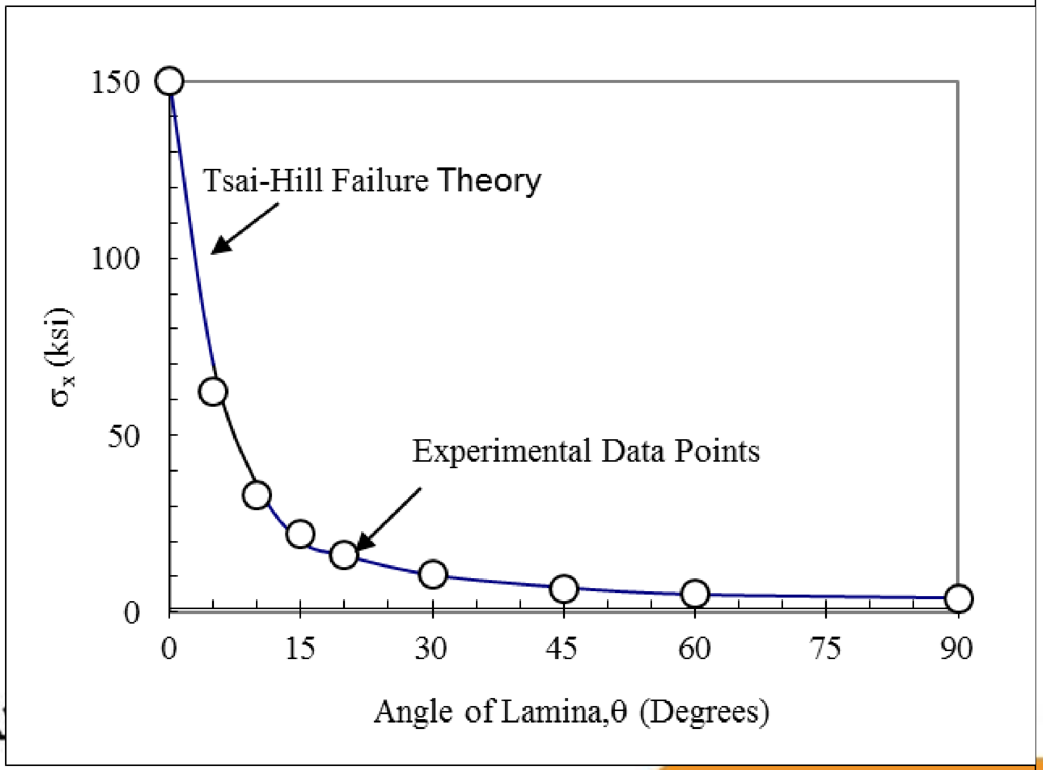


FIGURE 4.4
Maximum normal tensile stress in x-direction as function of angle of lamina using Tsai-Hill failure theory

Experimental Results and Tsai-Wu Failure Theory

$$H_1\sigma_1 + H_2\sigma_2 + H_6\tau_{12} + H_{11}\sigma_1^2 + H_{22}\sigma_2^2 + H_{66}\tau_{12}^2 + 2H_{12}\sigma_1\sigma_2 < 1$$

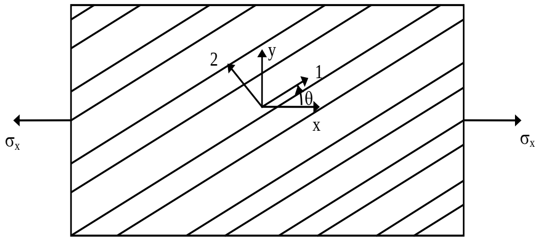
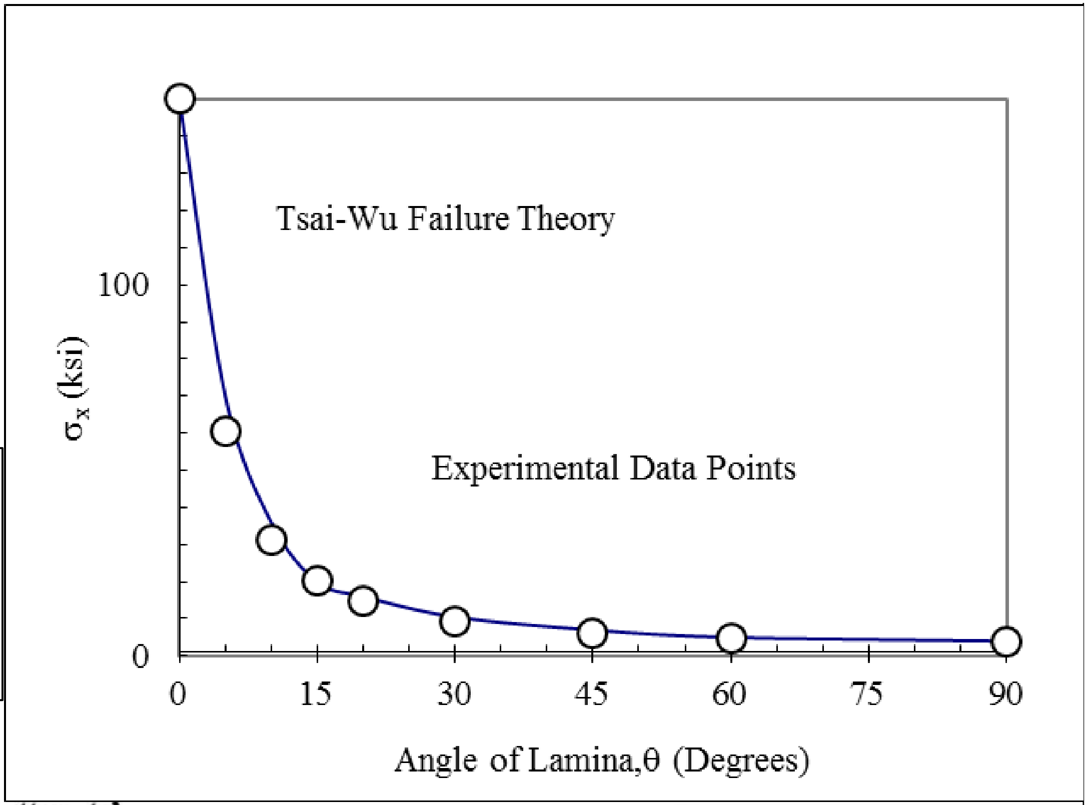
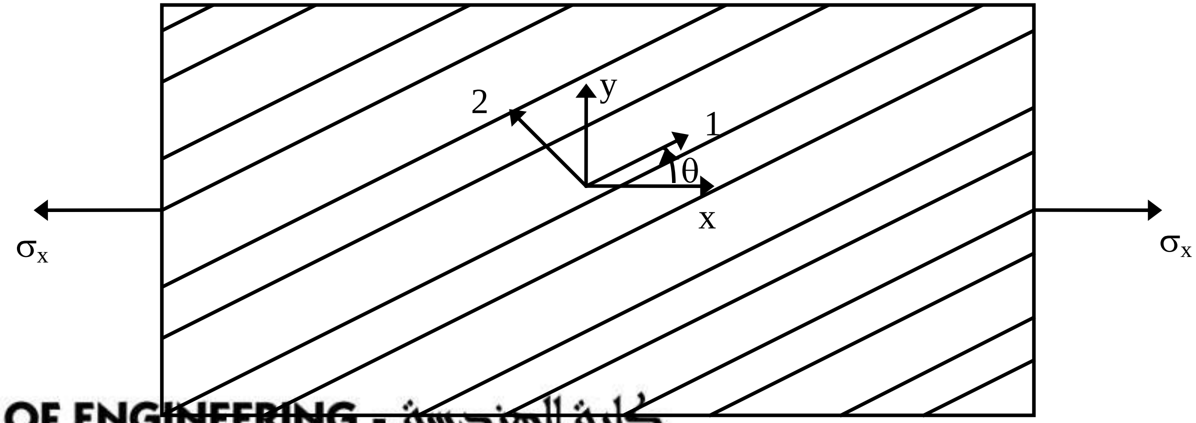


FIGURE 4.5
Maximum normal tensile stress in x-direction as function of angle of lamina using Tsai-Wu failure theory



Comparison of Strength Ratios

S = 16.33 (Maximum Stress failure theory),
= 16.33 (Maximum Strain failure theory),
= 10.94 (Tsai-Hill failure theory),
= 16.06 (Modified Tsai-Hill failure theory),
= 22.39 (Tsai-Wu failure theory)



Hygrothermal Stress-Strain Relationship

- For a unidirectional lamina

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} + \begin{bmatrix} \varepsilon_1^T \\ \varepsilon_2^T \\ 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_1^C \\ \varepsilon_2^C \\ 0 \end{bmatrix}$$

- Thermally induced strains:

$$\begin{bmatrix} \varepsilon_1^T \\ \varepsilon_2^T \\ 0 \end{bmatrix} = \Delta T \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_1^C \\ \varepsilon_2^C \\ 0 \end{bmatrix} = \Delta C \begin{bmatrix} \beta_1 \\ \beta_2 \\ 0 \end{bmatrix}$$

- Moisture induced strains:

Hygrothermal Stress-Strain Relationship

- For a unidirectional lamina

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} + \begin{bmatrix} \varepsilon_1^T \\ \varepsilon_2^T \\ 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_1^C \\ \varepsilon_2^C \\ 0 \end{bmatrix}$$

⬇

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 - \varepsilon_1^T - \varepsilon_1^C \\ \varepsilon_2 - \varepsilon_2^T - \varepsilon_2^C \\ \gamma_{12} \end{bmatrix}$$

Hygrothermal Stress-Strain Relationship

- For an angular lamina

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} + \begin{bmatrix} \varepsilon_x^T \\ \varepsilon_y^T \\ \gamma_{xy}^T \end{bmatrix} + \begin{bmatrix} \varepsilon_x^C \\ \varepsilon_y^C \\ \gamma_{xy}^C \end{bmatrix} \qquad (2.178)$$

- Thermally induced strains:

$$\begin{bmatrix} \varepsilon_x^T \\ \varepsilon_y^T \\ \gamma_{xy}^T \end{bmatrix} = \Delta T \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix} \qquad (2.179)$$

- Moisture induced strains:

$$\begin{bmatrix} \varepsilon_x^C \\ \varepsilon_y^C \\ \gamma_{xy}^C \end{bmatrix} = \Delta C \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{bmatrix} \qquad (2.180)$$

Transformation of CTE

- For an angular lamina

$$\begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy}/2 \end{bmatrix} = [T]^{-1} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix} \qquad (2.181)$$

$$[T]^{-1} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2 - s^2 \end{bmatrix} \qquad (2.95)$$

$$[T] = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \qquad (2.96)$$

$$c = \text{Cos}(\theta)$$

$$s = \text{Sin}(\theta) \qquad (2.97a,b)$$

Transformation of Coefficients of Moisture Expansion

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- For an angular lamina

$$\begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy}/2 \end{bmatrix} = [T]^{-1} \begin{bmatrix} \beta_1 \\ \beta_2 \\ 0 \end{bmatrix} \quad (2.182)$$

$$[T]^{-1} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2 - s^2 \end{bmatrix} \quad (2.95)$$

$$[T] = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \quad (2.96)$$

$$c = \mathrm{Cos}(\theta)$$

$$s = \mathrm{Sin}(\theta) \quad (2.97a,b)$$

Example 4.4

Find the following for a 60^0 angle lamina of Glass/Epoxy

a) coefficients of thermal expansion,

b) coefficients of moisture expansion,

c) strains under a temperature change of -100^0C and a moisture absorption of 0.02 kg/kg.

Use properties of unidirectional Glass/Epoxy lamina from Table 2.1.

Example 4.4

a) From Table 2.1,

$$\alpha_1 = 8.6\times10^{-6}\ m/m/{}^0C,$$

$$\alpha_2 = 22.1\times10^{-6}\ m/m/{}^0C.$$

$$\begin{bmatrix}\alpha_x\\ \alpha_y\\ \alpha_{xy}/2\end{bmatrix} = [T]^{-1}\begin{bmatrix}\alpha_1\\ \alpha_2\\ 0\end{bmatrix}. \qquad (2.181)$$

Using Equation (2.181), gives

$$\begin{bmatrix}\alpha_x\\ \alpha_y\\ \alpha_{xy}/2\end{bmatrix} = \begin{bmatrix}0.2500 & 0.7500 & -0.8660\\ 0.7500 & 0.2500 & 0.8660\\ 0.4330 & -0.4330 & -0.5000\end{bmatrix}\begin{bmatrix}8.6\times10^{-6}\\ 22.1\times10^{-6}\\ 0\end{bmatrix},$$

$$\begin{bmatrix}\alpha_x\\ \alpha_y\\ \alpha_{xy}\end{bmatrix} = \begin{bmatrix}18.73\times10^{-6}\\ 11.98\times10^{-6}\\ -11.69\times10^{-6}\end{bmatrix} m/m/{}^0C.$$

Example 4.4

b) From Table 2.1

$$\beta_1 = 0 \quad m/m/kg/kg,$$

$$\beta_2 = 0.6 \quad m/m/kg/kg.$$

$$\begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy}/2 \end{bmatrix} = [T]^{-1} \begin{bmatrix} \beta_1 \\ \beta_2 \\ 0 \end{bmatrix} \qquad (2.182)$$

Using Equation (2.182) gives

$$\begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy}/2 \end{bmatrix} = \begin{bmatrix} 0.2500 & 0.7500 & -0.8660 \\ 0.7500 & 0.2500 & 0.8660 \\ 0.4330 & -0.4330 & -0.5000 \end{bmatrix} \begin{bmatrix} 0.0 \\ 0.6 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{bmatrix} = \begin{bmatrix} 0.4500 \\ 0.1500 \\ -0.5196 \end{bmatrix} m/m/kg/kg$$

Example 4.4

c) Now using Equations (2.179) and (2.180) to calculate the strains as.

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 18.73\times10^{-6} \\ 11.98\times10^{-6} \\ -11.69\times10^{-6} \end{bmatrix}(-100) + \begin{bmatrix} 0.4500 \\ 0.1500 \\ -0.5196 \end{bmatrix}(0.02)$$

$$= \begin{bmatrix} 0.7127\times10^{2} \\ 0.1802\times10^{2} \\ -0.9223\times10^{2} \end{bmatrix} m/m$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} + \begin{bmatrix} \varepsilon_x^T \\ \varepsilon_y^T \\ \gamma_{xy}^T \end{bmatrix} + \begin{bmatrix} \varepsilon_x^C \\ \varepsilon_y^C \\ \gamma_{xy}^C \end{bmatrix} \quad (2.178$$

END